Linear Algebra and its Applications 317 (2000) 1–12
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Algebraic constructions of the minimal forbidden digraphs of strong sign nonsingular matrices ☆

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Received 16 April 1999; accepted 8 March 2000

Submitted by R.A. Brualdi

Abstract

A square real matrix A is called a strong sign nonsingular matrix (S^2NS matrix) if all the matrices with the same sign pattern as A are nonsingular and all the inverses of these matrices have the same sign pattern. S^2NS digraphs are digraphs associated with those S^2NS matrices with negative main diagonals. In this paper, we define the associated linear system of equations $L(D)$ (over the finite field F_2) for each digraph D, and then define an undirected graph $G(L(D))$ representing certain relations between the equations of $L(D)$. We obtain algebraic criteria to recognize the minimal forbidden configurations of S^2NS digraphs in terms of the solvability of the linear system $L(D)$ and some of its subsystems and the connectedness of the undirected graph $G(L(D))$. These algebraic criteria together with a conjunction operation of digraphs can be used to construct infinitely many new minimal forbidden configurations.

Keywords: Sign; Matrix; Directed graph

1. Introduction

The sign of a real number a is defined to be $1, -1$ or 0 according to the cases $a > 0,\ a < 0$ or $a = 0$, respectively.

☆ Research supported by Shanghai Science and Technology Developing Foundation 97XD14008 and National Natural Science Foundation of China 19831050.

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The sign pattern of a real matrix A is the $(0, 1, -1)$-matrix obtained from A by replacing each entry by its sign. The set of real matrices with the same pattern as A is called the qualitative class of A, and is denoted by $Q(A)$.

A square real matrix A is called a sign nonsingular (SNS) matrix, if each matrix with the same sign pattern as A is nonsingular. An SNS matrix A is called a strong sign nonsingular matrix (or $\mathrm{S^2NS}$ matrix), if the inverses of the matrices in $Q(A)$ all have the same sign pattern.

$\mathrm{S^2NS}$ matrices are extensively studied in qualitative matrix theory (which involves the study of "qualitative properties" which depend only on the sign patterns of the matrices, and has important applications in economics), see e.g., [1,2].

The study of $\mathrm{S^2NS}$ matrices are closely related to the study of certain class of digraphs. This is because that for a square real matrix A, the property of being an $\mathrm{S^2NS}$ matrix depends only on the sign pattern of A, while the sign pattern of A can be completely described by a signed digraph associated with A. We explain this relation more precisely in the following.

A signed digraph S is a digraph where each arc of S is assigned a sign +1 or -1 (the signed digraphs considered in this paper are assumed to contain no loops). The sign of a subdigraph S_1 of S is defined to be the product of the signs of all the arcs of S_1, denoted by $\mathrm{sgn}(S_1)$.

Let $A = (a_{ij})$ be a square real matrix of order n. The associated digraph $D(A)$ of A is defined to be the digraph with the vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(i, j) \mid a_{ij} \neq 0,\ i \neq j\}$. The signed digraph $S(A)$ of A is obtained from $D(A)$ by assigning the sign of a_{ij} to each arc (i, j) in $D(A)$. Clearly, $D(A)$ completely determines the zero pattern of A, and $S(A)$ completely determines the sign pattern of A.

A signed digraph S is called an $\mathrm{S^2NS}$ signed digraph if S satisfies the following two conditions:

1.1. The sign of every cycle of S is negative.

1.2. Every pair of paths in S with the same initial vertex and the same terminal vertex have the same sign.

A digraph D is called an $\mathrm{S^2NS}$ underlying digraph (or simply $\mathrm{S^2NS}$ digraph) if the arcs of D can be suitably assigned the signs so that the resulting signed digraph is an $\mathrm{S^2NS}$ signed digraph.

A significant result about the relation between $\mathrm{S^2NS}$ matrices and digraphs is [1,2]: a square real matrix A with a negative main diagonal is an $\mathrm{S^2NS}$ matrix if and only if its signed digraph $S(A)$ is an $\mathrm{S^2NS}$ signed digraph. From this result we think that the study of $\mathrm{S^2NS}$ matrices is essentially equivalent to the study of $\mathrm{S^2NS}$ signed digraphs, and the study of the zero patterns of $\mathrm{S^2NS}$ matrices is essentially equivalent to the study of $\mathrm{S^2NS}$ digraphs.

We can also see from the definitions that any signed subdigraph of an $\mathrm{S^2NS}$ signed digraph is an $\mathrm{S^2NS}$ signed digraph, and so any subdigraph of an $\mathrm{S^2NS}$ digraph is still an $\mathrm{S^2NS}$ digraph.

A digraph which is not an $\mathrm{S^2NS}$ digraph is also called an "$\mathrm{S^2NS}$ forbidden configuration" (or simply "a forbidden configuration"). If D is a forbidden configuration, but any proper subdigraph of D is not a forbidden configuration, then D is called a "minimal forbidden configuration" (MFC).

It is clear that any forbidden configuration contains an MFC as its subdigraph.

Now we introduce some operations on digraphs which can preserve the property of being, or being not, S^2NS digraphs.

Let D be a digraph. A splitting on a vertex x of D is the digraph obtained from D by inserting a new vertex x_1, a new arc (x, x_1), and replacing each arc of D of the form (x, v) by the arc (x_1, v). A subdivision on an arc (u, v) of D is the digraph obtained from D by deleting the arc (u, v) and then inserting a new vertex u_1 and two new arcs (u, u_1) and (u_1, v). A pair of oppositely directed arcs (x, y) and (y, x) in D is called an (undirected) edge, denoted by $[x, y]$. An "even edge subdivision" on an edge $[x, y]$ of D is the digraph obtained from D by deleting the edge $[x, y]$ and then inserting an even number of new vertices $x_1, x_2, \dots, x_{2k}$ and the new edges $[x, x_1], [x_1, x_2], \dots, [x_{2k-1}, x_{2k}]$ and $[x_{2k}, y]$.

It is not difficult to verify from the definitions that if a digraph D_1 is a vertex splitting, or an arc subdivision, or an even edge subdivision of a digraph D, then D is an S^2NS digraph if and only if D_1 is. Also if D_1 is an MFC, then so is D.

In view of this, we call an MFC D a "basic MFC" if D is an MFC and D is not a vertex splitting, or an arc subdivision, or an even edge subdivision of other digraphs.

The reverse digraph D' of a digraph D is a digraph obtained by reversing the directions of all the arcs of D. It is easy to see that D is an S^2NS digraph (or an MFC) if and only if D' is.

The following are some examples of basic MFCs.

Example 1.1 [3]. Let D_3 be the digraph with three vertices v, x, y and four arcs (x, y), (y, x), (x, v) and (y, v). Then (it is not hard to verify that) D_3 is a basic MFC.

Example 1.2 [4]. Let $k \geqslant 2$ be a positive integer and $t_1, t_2, \dots, t_k$ be nonnegative integers. Let G_i be a doubly directed path of length t_i with two end vertices v_i and u_i $(i = 1, 2, \dots, k)$.

Take $D(t_1, \dots, t_k)$ to be a digraph (as in [4, Theorem 2.1]) obtained by adding to the disjoint union of $G_1, \dots, G_k$ the new vertices $y_1, y_2; x_1, x_2, \dots, x_k$ and the following new arcs:

$$(x_i, v_i) \quad (i = 1, \dots, k); \qquad (x_i, u_{i+1}) \quad (i \equiv 1, \dots, k (\bmod k))$$
$$(u_i, y_1) \quad (i = 1, \dots, k); \qquad (v_i, y_2) \quad (i = 1, \dots, k).$$

Then it is proved in [4] that the digraph $D(t_1, \dots, t_k)$ is an MFC in the cases that $k \geqslant 2$ and $t_1 + \dots + t_k$ is odd. It is a basic MFC if we further require that each $t_i \in \{0, 1, 2\}$ $(i = 1, \dots, k)$.

An interesting problem in the study of S^2NS matrices and S^2NS digraphs is of the characterizations of S^2NS digraphs, especially the characterizations of S^2NS

digraphs in terms of the forbidden subdigraphs. On the other hand, since a digraph is an S^2NS digraph if and only if it does not contain any MFC as its subdigraph, it follows that this characterization problem for S^2NS digraphs is in fact equivalent to the problem of finding all the MFCs. Up to now, this problem is unsolved. Thomassen [3] first constructed the basic MFCs D_3 and D_3' (see Example 1.1), and proved that a strongly connected digraph is an S^2NS digraph if and only if it does not contain D_3 or D_3' as its subdigraph. Brualdi and Shader [2, p. 188] constructed a new basic MFC. In [4], this basic MFC has been extended to a family of digraphs $D(t_1, \ldots, t_k)$ as given in Example 1.2.

In this paper, we introduce some algebraic methods to characterize and construct MFCs. We first define the “associated linear system” $L(D)$ (over the finite field F_2) for each digraph D. The property of being an MFC for D can be characterized in terms of the solvability of the linear system $L(D)$ and some of its subsystems. We then define the “equation connection (undirected) graph” $G(L(D))$ associated with the linear system $L(D)$, which indicates the existence of the common unknowns between the equations of $L(D)$. Using the algebraic and graph theoretical arguments, we obtain (in Theorem 2.2) a criterion for recognizing MFCs in terms of the connectedness of the undirected graph $G(L(D))$ under certain conditions on the linear system $L(D)$. Using this criterion we construct a new basic MFC different from those in Examples 1.1 and 1.2 (and their reverse digraphs). In Section 3, we construct infinitely many new basic MFCs by using this new basic MFC together with a “conjunction” operation of digraphs which can preserve certain algebraic conditions needed in Theorem 2.2.

2. The algebraic constructions of MFCs

In this section, we introduce some algebraic methods to study MFCs. We obtain algebraic criteria to recognize MFCs. Using these criteria we are able to construct new basic MFCs. First we define the “associated linear system” $L(D)$ for each digraph D.

Definition 2.1. Let D be a digraph, $C_1, \ldots, C_r$ be all the (directed) cycles of D, and $\{P_i, Q_i\}$ $(i = 1, \ldots, k)$ be all the pairs of paths in D with the same initial vertex and the same terminal vertex. Then the “associated linear system” $L(D)$ of D is defined to be the following linear system of equations over the finite field F_2 (the field with two elements $\{0, 1\}$):

$$L(D): \quad \begin{cases} x(P_1) + x(Q_1) = 0, \\ \quad \vdots \\ x(P_k) + x(Q_k) = 0, \\ x(C_1) = 1, \\ \quad \vdots \\ x(C_r) = 1, \end{cases} \tag{2.1}$$

where for each arc e of D, there is a corresponding unknown $x(e)$ in the linear system $L(D)$, and for each subdigraph D_0 of D, we use $x(D_0)$ to denote $\sum_{e\in E(D_0)} x(e)$.

An equation of $L(D)$ is said to "contain an unknown $x(e)$" if the coefficient of $x(e)$ in that equation is not zero.

For a digraph D, the property of being an S^{2}NS digraph, or an MFC, can now be determined by the associated linear system $L(D)$.

Theorem 2.1. *Let D be a digraph. Then we have*:

1. *D is an* S^{2}NS *digraph if and only if the associated linear system $L(D)$ is solvable* (*over the field F_2*).
2. *If D contains no isolated vertices, then D is an MFC if and only if $L(D)$ is unsolvable, and $L(D\backslash\{e\})$ is solvable for each arc e of D* (*here $L(D\backslash\{e\})$ is a linear subsystem of $L(D)$ consisting of those equations which do not contain the unknown $x(e)$*).

Proof. 1. Suppose that $L(D)$ is solvable. Let $\{x(e) = c(e) \mid e \in E(D)\}$ be a solution of $L(D)$. Using this solution we assign the sign $(-1)^{c(e)}$ to each arc e of D. Then it is easy to verify that the resulting signed digraph is an S^{2}NS signed digraph. So D is an S^{2}NS digraph.

Conversely, suppose S is an S^{2}NS signed digraph with D as its underlying digraph. We take $x(e)$ to be 0 or 1 in F_2 according as the sign of the arc e in S is 1 or -1. Then these values of $x(e)$ $(e \in E(D))$ form a solution of the linear system $L(D)$.

2. Follows directly from (1) and the definition of MFCs. □

Definition 2.2. Let $L(D)$ be the associated linear system of a digraph D. Then D is said to be "2-arc regular" if for each arc e of D, the corresponding unknown $x(e)$ is contained in exactly two equations of $L(D)$.

We will see in the following (Examples 2.1 and 2.2) that those basic MFCs we know up to now (such as those given in Examples 1.1 and 1.2) are all 2-arc regular. Also, the infinitely many new basic MFCs which we will construct in this paper are also 2-arc regular.

In order to use the linear system $L(D)$ to study MFCs, we further define an undirected graph $G(L(D))$ associated with the linear system $L(D)$. The connectedness of this graph $G(L(D))$ will be an essential condition for a 2-arc regular digraph D to be an MFC.

Definition 2.3. Let D be a digraph. The equation connection graph $G(L(D))$ of the associated linear system $L(D)$ is an undirected graph such that each equation of $L(D)$ corresponds to a vertex of $G(L(D))$, and there is an edge between two vertices u and v in $G(L(D))$ if and only if the two equations in $L(D)$ corresponding to u and v contain a common unknown.

Using the algebraic properties of the linear system $L(D)$ and the equation connection graph $G(L(D))$, we can now give a criterion for determining whether a 2-arc regular digraph D is an MFC.

Theorem 2.2. *Let D be a 2-arc regular digraph with no isolated vertices. Then D is an MFC if and only if D satisfies the following two conditions*:
1. *The number of cycles in D is odd.*
2. *The equation connection graph $G(L(D))$ is connected.*

Proof. Let $A(L(D))$ and $\tilde{A}(L(D))$ be the coefficient matrix and augmented matrix of the linear system $L(D)$, respectively. The 2-arc regularity of D implies that the sum of all rows of $A(L(D))$ is a zero vector (since $1 + 1 = 0$ in F_2).

Sufficiency. Statement (1) implies that the sum of the rows of $\tilde{A}(L(D))$ is $(0, 0, \ldots, 0, 1)$. Hence, the linear system $L(D)$ is unsolvable.

Next, we show that (2) implies that any proper subset of the set of rows of the matrix $A(L(D))$ is linearly independent. Suppose not, then there must exist a proper subset of the set of rows of $A(L(D))$ whose sum is a zero vector (since the only nonzero number in F_2 is 1). Let V_1 be the (proper) vertex subset of the graph $G(L(D))$ corresponding to these rows of $A(L(D))$. Then there is no edge in the graph $G(L(D))$ with one end vertex in V_1 and another end vertex not in V_1 by the 2-arc regularity of D. So the graph $G(L(D))$ is not connected, a contradiction.

Now any proper subset of the set of rows of the matrix $A(L(D))$ is linearly independent. So any proper linear subsystem of $L(D)$ is solvable. Thus, for any arc e of D, the linear system $L(D\backslash\{e\})$ is solvable. Combining this and the unsolvability of $L(D)$, we conclude by Theorem 2.1 that D is an MFC.

Necessity. Now assume that D is an MFC. Suppose to the contrary that (2) does not hold (i.e., $G(L(D))$ is not connected). Then $L(D)$ can be partitioned into two proper linear subsystems $L_1(D)$ and $L_2(D)$ containing no commom unknowns. Now D is an MFC, so Theorem 2.1 implies that the two linear systems $L_1(D)$ and $L_2(D)$ are both solvable. On the other hand, $L_1(D)$ and $L_2(D)$ contains no common unknowns, so the union of these two solvable linear systems $L_1(D)$ and $L_2(D)$, namely, the linear system $L(D)$, is also solvable, contradicting the fact that D is an MFC. So (2) holds.

By the same arguments as in the sufficiency part, (2) implies that any proper subset of the set of rows of the matrix $A(L(D))$ is linearly independent. So

$$\operatorname{rank}(A(L(D))) = \text{number of rows of } A(L(D)) - 1.$$

On the other hand, D is an MFC implies that $L(D)$ is unsolvable, so we have

$$\operatorname{rank}(\tilde{A}(L(D))) = \operatorname{rank}(A(L(D))) + 1 = \text{number of rows of } \tilde{A}(L(D))$$

so the rows of $\tilde{A}(L(D))$ are linearly independent, and thus the sum of the rows of $\tilde{A}(L(D))$ is not a zero vector. But the sum of the rows of $A(L(D))$ is a zero vector,

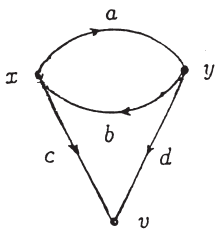

Fig. 1. The digraph D_3.

so the sum of the constant terms of all the equations of $L(D)$ is not zero. This implies that the number of cycles in D is odd, and so (1) holds.

This completes the proof of the theorem. □

We now verify that both the digraphs D_3 (in Example 1.1) and $D(t_1, \ldots, t_k)$ (in Example 1.2) satisfy the 2-arc regularity condition and the connectedness condition for $G(L(D))$ in Theorem 2.2. These properties of the digraphs $D(t_1, \ldots, t_k)$ will also be used in the constructions of the new basic MFCs in Section 3.

For convenience, the unknowns of the associated linear systems in the following examples are indicated directly in Figs. 1–4 near the corresponding arcs.

Example 2.1. $D = D_3$ (as defined in Example 1.1, also see Fig. 1). Then $L(D)$ is the following linear system:

$$L(D): \qquad \begin{cases} a + d + c = 0, \\ b + c + d = 0, \\ a + b = 1, \end{cases}$$

and $G(L(D))$ is the complete graph K_3. Clearly D is 2-arc regular and $G(L(D))$ is connected. So D_3 is an MFC by Theorem 2.2.

Example 2.2. $D = D(t_1, \ldots, t_k)$ (as defined in Example 1.2). For convenience, we first consider the special case $t_1 = \cdots = t_k = 0$ and denote $D(k) = D(0, 0, \ldots, 0)$ (see Fig. 2).

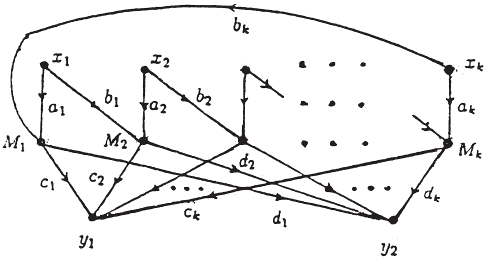

Fig. 2. The digraph $D(k)$.

Notice that for any vertex x_i $(i = 1, \ldots, k)$ and any vertex y_j $(j = 1, 2)$, there is exactly one pair of paths from x_i to y_j, and these $2k$ pairs of paths are all the pairs of paths in $D(k)$ with the same initial vertex and the same terminal vertex. So the associated linear system $L(D(k))$ is the following:

$$L(D(k)):\quad \left\{\begin{aligned} a_1 + b_1 + c_1 + c_2 &= 0,\\ a_2 + b_2 + c_2 + c_3 &= 0,\\ &\vdots\\ a_{k-1} + b_{k-1} + c_{k-1} + c_k &= 0,\\ a_k + b_k + c_k + c_1 &= 0,\\ a_1 + b_1 + d_1 + d_2 &= 0,\\ a_2 + b_2 + d_2 + d_3 &= 0,\\ &\vdots\\ a_{k-1} + b_{k-1} + d_{k-1} + d_k &= 0,\\ a_k + b_k + d_k + d_1 &= 0. \end{aligned}\right. \tag{2.2}$$

From (2.2) it can be easily checked that $D(k)$ is 2-arc regular. Also, the graph $G(L(D(k)))$ is a cylinder with a k-polygon as its base, so $G(L(D(k)))$ is connected.

For the general cases where $t_1, \ldots, t_k$ $(k \geqslant 2)$ are arbitrary nonnegative integers, it can be verified similarly that $D(t_1, \ldots, t_k)$ is 2-arc regular and the graph $G(L(D(t_1, \ldots, t_k)))$ is connected. (In fact, the graph $G((L(D(t_1, \ldots, t_k)))$ can be obtained by adding to the cylinder graph $G(L(D(k)))$ $(t_1 + \cdots + t_k)$ many new vertices and $2(t_1 + \cdots + t_k)$ new edges where for each new vertex u^*, there are exactly two new edges between u^* and some two old vertices in $G(L(D(k)))$.) Thus, the fact that the digraph $D(t_1, \ldots, t_k)$ is an MFC in the cases when $t_1 + \cdots + t_k$ is odd $(k \geqslant 2)$ [4] can now be verified by using Theorem 2.2.

Now we use Theorem 2.2 to construct a new basic MFC.

Example 2.3. Let D be the digraph as in Fig. 3.

Notice that except for the cases $(i, j) = (1, 3)$ and $(i, j) = (2, 2)$, there is exactly one pair of paths from the vertex u_i to the vertex w_j $(1 \leqslant i, j \leqslant 3)$, and these seven

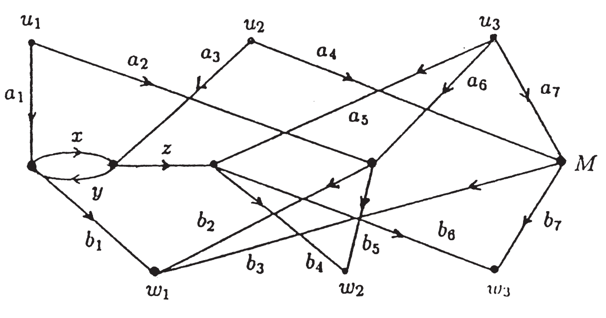

Fig. 3.

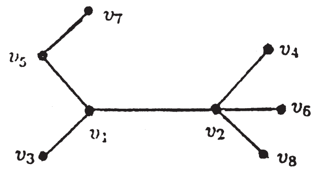

Fig. 4.

pairs of paths are all the pairs of paths in D with the same initial vertex and the same terminal vertex. So $L(D)$ is the following linear system:

$$L(D): \quad \begin{cases} a_1+b_1+a_2+b_2=0, \\ a_1+x+z+b_4+a_2+b_5=0, \\ a_3+y+b_1+a_4+b_3=0, \\ a_3+z+b_6+a_4+b_7=0, \\ a_6+b_2+a_7+b_3=0, \\ a_5+b_4+a_6+b_5=0, \\ a_5+b_6+a_7+b_7=0, \\ x+y=1. \end{cases} \tag{2.3}$$

From (2.3) we see that this digraph D is 2-arc regular. Also, the equation connection graph $G(L(D))$ contains the following connected spanning subgraph (in Fig. 4), so $G(L(D))$ is connected.

By using Theorem 2.2 we conclude that D is an MFC. Since D contains no vertex with indegree one and no vertex with outdegree one and D contains only one cycle, we see that D is a basic MFC.

In the next section, we construct infinitely many new basic MFCs by using Theorem 2.2 and a "conjunction" operation of digraphs defined in Definition 3.1.

3. MFCs and the conjunction of digraphs

In this section, we will use a conjunction operation of digraphs to construct new MFCs from the old MFCs. First we need some definitions.

A (strong) component of a digraph D is called a trivial component if it contains no arcs. A component H of D is an initial (or terminal) component if there is no arc of D from a vertex outside H (or in H) to a vertex in H (or outside H). An intermediate component is a component which is neither an initial component nor a terminal component.

A subdigraph G of a digraph D is called a **Y**-type subdigraph if G is a digraph as in Fig. 5, where A and B are trivial initial components of D, T is a trivial terminal component of D and M is a trivial intermediate component of D such that there is no arc in D from A to T or from B to T, and there is no arc in D between M and other initial or intermediate components of D. We denote this **Y**-type subdigraph G by $G=D(A,B,M,T)$.

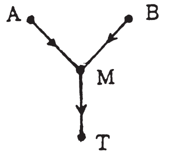

Fig. 5.

Definition 3.1. Let D_1 and D_2 be two digraphs. $G_i = D_i(A_i, B_i, M_i, T_i)$ be a **Y**-type subdigraph of D_i $(i = 1, 2)$, write the arcs

$$c_i = (A_i, M_i), \qquad d_i = (B_i, M_i), \qquad e_i = (M_i, T_i), \quad (i = 1, 2).$$

The conjunction digraph of D_1 and D_2 on G_1 and G_2, denoted by $D_1(G_1) * D_2(G_2)$ (or simply $D_1 * D_2$), is a digraph obtained by identifying G_1 and G_2 (i.e., identifying the corresponding vertices and arcs in G_1 and G_2) and then deleting the arc $e_1 = e_2$ in the resulting digraph.

For convenience, the new vertices (and arcs) in $D_1 * D_2$ obtained from identifying old vertices (and arcs) in G_1 and G_2 are denoted by A, B, M, T (and c, d), respectively.

It is obvious that in the conjunction digraph $D_1 * D_2$, there is no arc from a component in $D_1 \backslash G_1$ to a component in $D_2 \backslash G_2$. It follows from this observation that there is also no path from a component H_1 in $D_1 \backslash G_1$ to a component H_2 in $D_2 \backslash G_2$ (and vice versa), for otherwise this path must contain an arc from some component F_1 in $D_1 \backslash G_1$ to some component F_2 in $D_2 \backslash G_2$. This means that any path in $D_1 * D_2$ is either entirely contained in D_1 or entirely contained in D_2.

Theorem 3.1. *Let $D_1 * D_2$ be the conjunction digraph of D_1 and D_2 on their* **Y***-type subdigraphs $G_1 = D_1(A_1, B_1, M_1, T_1)$ and $G_2 = D_2(A_2, B_2, M_2, T_2)$. Then we have*:

1. *If D_1 and D_2 are both 2-arc regular, then so is $D_1 * D_2$.*
2. *If D_1 and D_2 are both 2-arc regular, and the equation connection graphs $G(L(D_1))$ and $G(L(D_2 \backslash \{e_2\}))$ are both connected, then $G(L(D_1 * D_2))$ is also connected.*

Proof. 1. We first consider the associated linear system $L(D_i)$ of the digraph D_i $(i = 1, 2)$. There are two equations in $L(D_i)$ containing the unknown $x(e_i)$ by the 2-arc regularity of D_i. Now the only two paths in D_i containing the arc e_i (except e_i itself) are $c_i + e_i$ and $d_i + e_i$, so $L(D_i)$ must be of the following form $(i = 1, 2)$:

$$L(D_i): \qquad \begin{cases} L(D_i \backslash \{e_i\}), \\ x(c_i) + x(e_i) + x(P_i) = 0, \\ x(d_i) + x(e_i) + x(Q_i) = 0, \end{cases} \tag{3.1}$$

where P_i (and Q_i) is the unique path in D_i from A_i (and B_i) to T_i different from $c_i + e_i$ (and $d_i + e_i$).

Now we consider the linear system $L(D_1 * D_2)$. Suppose that R_1 and R_2 are two paths in $D_1 * D_2$ with the same initial vertex and the same terminal vertex which are neither both in D_1 nor both in D_2. Then we may assume that R_i passes through some component H_i in $D_i \backslash G_i$ $(i = 1, 2)$. It follows from the comments before this theorem that the common initial vertex of R_1 and R_2 must be A or B, and the common terminal vertex of R_1 and R_2 must be T. Therefore, we must have

$$\{R_1, R_2\} = \{P_1, P_2\} \quad \text{or} \quad \{R_1, R_2\} = \{Q_1, Q_2\}.$$

Also we notice that any cycle of $D_1 * D_2$ is either a cycle of D_1 or a cycle of D_2. So we conclude that the linear system $L(D_1 * D_2)$ have the following form:

$$L(D_1 * D_2): \qquad \begin{cases} L(D_1\backslash\{e_1\}), \\ x(P_1) + x(P_2) = 0, \\ x(Q_1) + x(Q_2) = 0, \\ L(D_2\backslash\{e_2\}). \end{cases} \tag{3.2}$$

From (3.2) it is easy to see that $L(D_1 * D_2)$ is 2-arc regular.

2. In the form (3.2) of the linear system $L(D_1 * D_2)$, let V_1 and V_2 be the vertex subsets of the graph $G(L(D_1 * D_2))$ corresponding to the subsystems $L(D_1\backslash\{e_1\})$ and $L(D_2\backslash\{e_2\})$; let u and v be the vertices of the graph $G(L(D_1 * D_2))$ corresponding to the equations $x(P_1) + x(P_2) = 0$ and $x(Q_1) + x(Q_2) = 0$, respectively. By the assumptions, the graph $G(L(D_2\backslash\{e_2\}))$ is connected, so the vertices of V_2 are all contained in one component of $G(L(D_1 * D_2))$. Also $L(D_2\backslash\{e_2\})$ contains some unknown in $x(P_2)$ and some unknown in $x(Q_2)$ by (3.1), so the vertices u, v and the vertices of V_2 are all contained in one component of $G(L(D_1 * D_2))$.

Now suppose that $G(L(D_1 * D_2))$ is not connected. Then the set of equations in the subsystem $L(D_1\backslash\{e_1\})$ can be divided into two parts $L_1(D_1\backslash\{e_1\})$ and $L_2(D_1\backslash\{e_1\})$ such that $L_1(D_1\backslash\{e_1\})$ is nonempty (contains at least one equation) and $L_1(D_1\backslash\{e_1\})$ contains no common unknowns with all the other equations in $L(D_1 * D_2)$. It follows that $L_1(D_1\backslash\{e_1\})$ does not contain the unknowns in $x(P_1)$ and $x(Q_1)$, and does not contain the unknowns $x(c_1)$ and $x(d_1)$ (since $L(D_2\backslash\{e_2\})$ contains $x(c_2)$ and $x(d_2)$, and $c_1 = c_2$, $d_1 = d_2$ in $D_1 * D_2$). Thus, $L_1(D_1\backslash\{e_1\})$ contains no common unknowns with all the other equations in $L(D_1)$ (see (3.1)), and so the graph $G(L(D_1))$ is not connected, a contradiction. So $G(L(D_1 * D_2))$ is connected. □

We now use Theorem 3.1 to construct a family of infinitely many new basic MFCs. Take D_1 to be the digraph in Example 2.3. Take a **Y**-type subdigraph G_1 of D_1 as $G_1 = D_1(u_2, u_3, M, w_3)$, (see Fig. 3). Take D_2 to be the digraph $D(k) = D(0, 0, \ldots, 0)$ in Example 2.2 (where $D(t_1, \ldots, t_k)$ is the digraph as defined in Example 1.2). Take a **Y**-type subdigraph G_2 of D_2 as $G_2 = D_2(x_1, x_2, M_2, y_1)$ (see Fig. 2), and take the conjunction digraph $D_1 * D_2 = D_1(G_1) * D_2(G_2)$. From Examples 2.2 and 2.3 we know that both D_1 and D_2 are 2-arc regular, so $D_1 * D_2$ is also 2-arc regular by Theorem 3.1. Also $G(L(D_1))$ is connected by Example 2.3. Write the arc $e_2 = (M_2, y_1)$ in G_2. Then the graph $G(L(D_2\backslash\{e_2\}))$ is a subgraph of $G(L(D(k)))$ obtained by deleting two adjacent vertices of $G(L(D(k)))$, so it is not hard to verify that the graph $G(L(D_2\backslash\{e_2\}))$ is also connected (in fact, the cylinder graph $G(L(D(k)))$ is 3-connected for $k \geqslant 3$). It follows from Theorem 3.1 that $G(L(D_1 * D_2))$ is also connected. Now clearly $D_1 * D_2$ contains no isolated vertices and contains exactly one cycle, so $D_1 * D_2$ is an MFC by Theorem 2.2. Finally, since $D_1 * D_2$ contains no vertex with indegree one and no vertex with outdegree one, we conclude that $D_1 * D_2$ is a basic MFC.

By taking $k = 2, 3, \ldots,$ for $D_2 = D(k)$, the conjunction digraphs $D_1 * D(k)$ actually give an infinite family of new basic MFCs.

References

[1] L. Bassett, J. Maybee, J. Quirk, Qualitative economics and the scope of the correspondence principle, Econometrica 36 (1968) 544–563.

[2] R.A. Brualdi, B.L. Shader, Matrices of Sign-solvable Linear System, Cambridge University Press, Cambridge, MA, 1995.

[3] C. Thomassen, When the sign pattern of a square matrix determines uniquely the sign pattern of its inverse, Linear Algebra Appl. 119 (1989) 27–34.

[4] J.-Y. Shao, On digraphs and forbidden configurations of strong sign nonsingular matrices, Linear Algebra Appl. 282 (1998) 221–232.